Amplitude equation at ionization instability and the onset of turbulence in a neon glow discharge

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We investigate the bifurcations of ionization waves from the homogeneous stationary state of the positive column in a neon glow discharge. In the weak nonlinear region the wave dynamics is approximated by an amplitude equation of the Ginzburg-Landau type with complex coefficients and an additional integral term. This nonlocal term describes the influence of the external circuit on the ionization waves. Conditions are derived for the band of Eckhaus stable wave solutions. The dependence of the complex coefficients on the plasma parameters is discussed and used to classify the solution manifold of the amplitude equation. Raising the gas pressure from 130 Pa up to 720 Pa, the intermittency, bichaos, and amplitude turbulence regions are visited successively. Some of these theoretical results are supported by numerical calculations for selected parameter values. [S1063-651X(98)13309-3]

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I. INTRODUCTION

Driven by external forces, a spatially extended system very often undergoes a transition from a uniform state to complex spatiotemporal behavior, i.e., one observes stationary periodic patterns, wave excitation, and different forms of spatiotemporal chaos [1]. Such behavior is also observed in the positive column of noble gas discharges, e.g., in neon discharges, if the discharge current I is varied. These pattern formations can be explained by the complicated interplay of diffusion, convection, conduction, production, and loss processes. Mostly the nonlinear dynamics of ionization waves is investigated on the basis of a hydrodynamical model, which also includes the effect of the axial electric field [2]. To obtain satisfactory agreement with the experimental results, not only the dynamics of the charge carriers but also the influence of excited atoms has to be taken into account. Furthermore, the external circuit characterized by a voltage U and an Ohmic resistance R_a has to be considered and this yields a global coupling term.

It is well known that in neon discharges also turbulence may be observed in certain parameter regions [3,4]. Recent measurements [5] have shown that the transition to turbulence may also appear near the stability boundary of the homogeneous equilibrium. This motivates us to use modern methods of bifurcation theory, which properly work near the critical parameter values, where the instability sets in.

In two recent papers [6,7] we have systematically investigated the bifurcations developing from the homogeneous column and their dependence on the strength of the global coupling term. By changing the Ohmic resistance we can realize supercritical and subcritical Hopf bifurcations as well as generalized Hopf bifurcations. Moreover, different types of Hopf-Hopf bifurcations [8] can be shown to exist. The corresponding calculations use center manifold and normal form theories and describe the instability of one and two modes, respectively. These methods are well suited for the short positive column. In the case of a long column, where many modes become unstable simultaneously near the critical current, the description by an equation of the Ginzburg-Landau type becomes relevant. The complex coefficients of this equation can by calculated from the basic equations using multiple scale techniques. For ionization waves near the Pupp critical current [9] Bekki [10] derived an amplitude equation in the form of a nonlinear Schrödinger equation.

In our case of global coupling the usual complex Ginzburg-Landau equation (CGLE) is supplemented by a third-order integral term

$$\frac{\partial A}{\partial t}=\mu A-v_g\frac{\partial A}{\partial z}+b\frac{\partial^2 A}{\partial z^2}+cA^*A^2+g\frac{A}{l}\int_0^l A^*A\,dz, \tag{1.1}$$

with the real bifurcation parameter μ, the group velocity v_g, and complex coefficients b,c,g. In the case of real b,c,g Eq. (1.1) was discussed by Elmer [11]. Equation (1.1) describes the slow spatiotemporal development of the complex amplitude A of a plane wave near a supercritical Hopf bifurcation. Qualitatively, the same types of solutions exist as in the simple CGLE. Nevertheless, the modifications caused by the global coupling are described.

Currently, there is a growing interest on the influence of global coupling on spatiotemporal pattern formation. Dynamical systems ranging from gas discharges [12] and semiconductor systems [13] to chemical [14,15] and biological [16] systems have been investigated. To study the effect of global coupling various models such as coupled map lattices [17], coupled Rössler oscillators [18], reaction-diffusion systems [19], and extended Ginzburg-Landau systems [20] were used. Specific phenomena related to global coupling in oscillatory systems are the breakdown of synchronization caused by defects, the spontaneous formation of phase domains, and the development of standing waves.

This paper is organized as follows. In Sec. II the basic equations are described and briefly discussed. Section III contains the derivation of the amplitude equation. Some results concerning plane waves and their stability are discussed in Sec. IV. In particular the modifications resulting from global coupling are emphasized. Results that describe the ionization instability in a low-pressure neon discharge are discussed in Sec. V. The dependence of the parameters b,c,g on the gas pressure is calculated also in Sec. V. Furthermore, the correctness of the predictions with respect to ordered and turbulent behavior, respectively, is verified by numerical calculations.

II. BASIC EQUATIONS

The physical model equations we are using in this paper are based on a hydrodynamic description that takes into account the equations of continuity for ions, electrons, metastable atoms, and the electron energy balance equation. This system is completed by Maxwell's equations describing the quasistatic electric field. In order to make this set of equations analytically tractable, some physically motivated simplifying assumptions are used. Now we briefly sketch some basic approximations; however, the complete discussion can be found in [6].

The assumption of quasineutrality reduces the number of dependent variables by one. This assumption is justified because the wavelength of the considered ionization phenomena is much larger than the Debye radius of the plasmas concerned. Motivated by experimental results in cylindrical tubes, we describe a discharge with cylindrical symmetry and therefore all dependent quantities are functions of the axial coordinate z and the radius. By means of an averaging with respect to the radial dependence our problem becomes a one-dimensional one. This averaging is based on the assumption of a radial dependence of the densities in terms of a zeroth-order Bessel function. After a rescaling of the basic equations in terms of dimensionless variables, the time derivative of the electron energy balance equation is multiplied by a small factor and is also neglected. Furthermore, we take into consideration the external current circuit by the balance equation

$$U=R_aI(t)+\int_0^L E(z,t)dz, \tag{2.1}$$

where R_a is the external resistance, L denotes the length of the positive column, $E(z,t)$ is the axial component of the electric field, and $I(t)$ is the total current.

In the next step one considers the homogeneous and stationary solutions that describe the equilibrium states of the positive column [6]. Indeed such solutions exist for the system of balance equations by compensating for the different production and loss terms. Let $N(z,t)$, $M(z,t)$, $T(z,t)$, and $E(z,t)$ be the radial averaged densities of ions, metastable atoms, electron temperature, and the axial component of the electric field, respectively. The equilibrium state is designated by N_0,M_0,T_0,E_0,I_0. We introduce the relative deviations from the equilibrium state

$$u=\frac{N-N_0}{N_0},\quad m=\frac{M-M_0}{M_0},\quad v=\frac{T-T_0}{T_0},\quad w=\frac{E-E_0}{E_0},\quad j=\frac{I-I_0}{I_0} \tag{2.2}$$

and define the dimensionless independent variables

$$z'=\frac{E_0}{T_0}z,\quad t'=b_i\frac{E_0^2}{T_0}t,$$

where T_0 is measured in units of volts and b_i denotes the mobility of the ions. Furthermore, we omit the prime on the variables z',t' hereafter for notational convenience. Then, making an expansion of the nonlinear terms up to the third order with respect to all field variables, the set of basic equations obtains the principal structure

$$\frac{\partial}{\partial t}(\hat{T}\vec{X})=\hat{L}\vec{X}+\vec{N_2}(X,X)+\vec{N_3}(X,X,X)+\cdots, \tag{2.3}$$

where $\vec{X}$ denotes the field vector and $\hat{T}$ is a projection operator

$$\vec{X}=\begin{pmatrix} u\\ m\\ v\\ w\end{pmatrix},\quad \hat{T}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Moreover, $\hat{L}\vec{X}$ indicates the linear part

$$\hat{L}\vec{X}=\begin{pmatrix} \alpha\frac{\partial^2u}{\partial z^2}+\eta_1u+\eta_3m+\beta\frac{\partial^2v}{\partial z^2}+\eta_2v\\ D\frac{\partial^2m}{\partial z^2}+\eta_4u+\eta_6m+\eta_5v\\ -\delta_1\frac{\partial^2u}{\partial z^2}-u-h_4m-\delta_2\frac{\partial^2v}{\partial z^2}+\kappa\frac{\partial v}{\partial z}-h_1v+w+j\\ \alpha\frac{\partial u}{\partial z}+u+\beta\frac{\partial v}{\partial z}+w-j\end{pmatrix} \tag{2.4}$$

and N_2,N_3 stand for the quadratic and cubic nonlinearities, respectively, which are given by

$$\vec{N_2}=\begin{pmatrix} \alpha v\frac{\partial^2u}{\partial z^2}+(\alpha+\beta)\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}+\beta u\frac{\partial^2v}{\partial z^2}+\rho_1uv+\rho_2v^2+\rho_3um+\rho_4vm+\rho_5m^2\\ \rho_6uv+\rho_7v^2+\rho_8um+\rho_9vm+\rho_{10}m^2\\ -2\delta_1v\frac{\partial^2u}{\partial z^2}-\delta_2(u+v)\frac{\partial^2v}{\partial z^2}-\delta_3\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}-\delta_2\left(\frac{\partial v}{\partial z}\right)^2+jw+\kappa j\frac{\partial v}{\partial z}-h_1uv\\ -h_2v^2-h_4um-h_5vm\\ \alpha v\frac{\partial u}{\partial z}+\beta u\frac{\partial v}{\partial z}+uw\end{pmatrix}$$

and

$$\vec{N_3}=\begin{pmatrix} \sigma_0 umv+\sigma_1 uv^2+\sigma_2 v^3+\sigma_3 mv^2+\sigma_4 vm^2 \\ \sigma_5 uv^2+\sigma_6 v^3+\sigma_7 mv^2+\sigma_8 vm^2 \\ -\delta_1 v^2\frac{\partial^2 u}{\partial z^2}-\delta_2 uv\frac{\partial^2 v}{\partial z^2}-\delta_3 v\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}-\delta_2 u\left(\frac{\partial v}{\partial z}\right)^2-h_2 uv^2-h_3 v^3 \\ -h_5 uvm-h_6 mv^2 \\ 0. \end{pmatrix}$$

Taking into account the definition of the current j [6],

$$j=-\frac{R}{R_a l}\int_0^l w\, dz, \tag{2.5}$$

where $l=LE_0/T_0$ is the dimensionless length of the discharge and $R=E_0 L/I_0$ its equilibrium resistance, Eq. (2.3) is a coupled system of partial integro-differential equations. η_k, h_l, σ_m, and ρ_n are coefficients that result from the series expansion of the production and loss terms up to third order (cf. [6,7]). In contrast to the kinetic coefficients $\alpha,\beta,\delta_j,\kappa$, these parameters depend on the actual equilibrium solution N_0,M_0,T_0,E_0. The solutions of Eq. (2.3) also depend on the boundary conditions at the ends of the positive column. From the experimental point of view it is not quite clear what boundary conditions have to be chosen to approximate a real positive column. On the other hand, our results in [6,7] have shown that the bifurcation behavior is almost independent of the special boundaries at least in the limit of long discharges. In this contribution we use periodic boundary conditions and it also has the advantage that all calculations are simplified.

In the first step we are interested in the stability properties of the equilibrium discharge, i.e., we consider the linearized system

$$\frac{\partial}{\partial t}(\hat{T}\vec{X})=\hat{L}\vec{X}. \tag{2.6}$$

A plane wave ansatz yields the dispersion relation that can be solved by means of numerical methods. Figure 1 shows a set of instability curves (or neutral curves) of the p waves parametrized by the pressure p_0. Below the actual curve the positive column is stable. Crossing the curve by increasing the discharge current I_0, a supercritical Hopf bifurcation takes place (cf. [7]) and one observes ionization waves. A characteristic property is the local minimum of the instability curve at a critical wave number k_c. More precisely, we have a two-dimensional manifold of local minima by variation of the pressure p_0 and the radius r_0 of the discharge. It is well known that the wave dynamics in the neighborhood of the local minimum can be well approximated by an amplitude equation of the Ginzburg-Landau type. In order to find such an equation, there are two essential assumptions: (a) the instability curve near the local minimum is approximated by a parabola and (b) the nonlinearity is given by the Hopf normal form equation, i.e., by a third-order term. In this sense, a Ginzburg-Landau–type amplitude equation represents the normal form of the wave dynamics in the weak nonlinear region.

III. DERIVATION OF AN AMPLITUDE EQUATION

Let the discharge current I_0 be the control parameter and I_c its critical value at the minimum of the instability curve. Then we shall consider a natural small parameter of the problem ε defined through

$$\varepsilon^2=\frac{I_0-I_c}{I_c}, \tag{3.1}$$

which is a measure of the distance from the bifurcation point. In order to find the amplitude equation, we use the method described in [21]. The wave vector $\vec{X}$ is expanded in a power series with respect to ε

$$\vec{X}=\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}\vec{X}^{(\alpha)}=\varepsilon\vec{X}^{(1)}+\varepsilon^2\vec{X}^{(2)}+\cdots \tag{3.2}$$

and, moreover, each of these terms is expanded in a Fourier series by

$$\vec{X}^{(\alpha)}=\sum_{n=-\infty}^{\infty}\vec{X}_n^{(\alpha)}(\tau,\xi)\exp\, i(k_n z-\omega_n t). \tag{3.3}$$

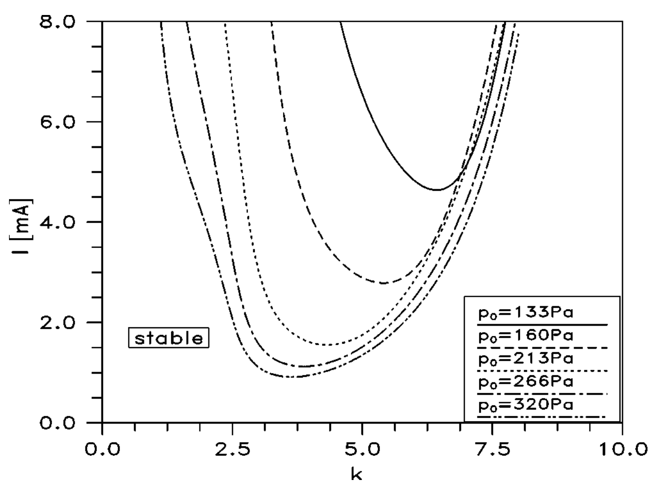

FIG. 1. Instability curves of p waves parametrized by selected pressure values (the radius $r_0=1$ cm and k is the dimensionless wave number).

Here $k_n = nk_1$ and $\omega_n = n\omega_1$ are multiples of the basic pair $(\omega_1, k_1) = (\omega_c, k_c)$, i.e., (ω_1, k_1) correspond to the critical mode that becomes unstable at the minimum of the instability curve. Furthermore, the Fourier coefficients $\vec{X}_n^{(\alpha)}$ represent weakly varying functions of space and time, i.e., they depend on the stretched variables

$$\tau = \varepsilon^2 t, \quad \xi = \varepsilon(z - ct), \tag{3.4}$$

where c is a free real parameter, which must be fixed consistently in the course of our calculations. Because the wave vector $\vec{X}$ is a real one, the coefficients $\vec{X}_n^{(\alpha)}$ have to fulfill the conditions

$$\vec{X}_n^{(\alpha)} = (\vec{X}_{-n}^{(\alpha)})^*, \tag{3.5}$$

where the asterisk denotes the conjugate complex. As the last step one has to take into consideration that the constants η_k, h_l, σ_m, ρ_n, D, and R depend on the actual discharge current I_0 (see [6] for their definitions), i.e., if we go into the instability region, these parameters depend on the actual value of ε. This dependence can be approximated by a Taylor expansion near the critical point (I_c, k_c)

$$\eta_k = \eta_k|_{0,k_c} + \eta_k'|_{0,k_c}\varepsilon^2 + 1/2\,\eta_k''|_{0,k_c}\varepsilon^4 + \cdots, \tag{3.6}$$

where $0,k_c$ indicates the critical point and the prime stands for the partial derivative with respect to ε^2. The same expansion is used also for the remaining parameters. Substituting all these expansions into the basic equation (2.3) and equating the coefficients of equal powers of ε leads to a hierarchy of linear inhomogeneous equations. The evaluations are straightforward but involve tedious algebra and therefore we report here only selected results.

A. First order of perturbation theory

Because the Fourier modes form an independent system of functions, the first order ε^1 splits into an infinite system of homogeneous algebraic equations, where the property (3.5) reduces the interesting mode number to $n \geqslant 0$. In the case $n = 1$ this system reads

$$\hat{\Omega}(k_1, \omega_1)\vec{X}_1^{(1)} = \vec{0}, \tag{3.7}$$

with the matrix operator

$$\hat{\Omega}(k_1,\omega_1) = \begin{pmatrix} \alpha k_1^2 - \eta_1 - i\omega_1 & -\eta_3 & \beta k_1^2 - \eta_2 & 0 \\ -\eta_4 & Dk_1^2 - \eta_6 - i\omega_1 & -\eta_5 & 0 \\ 1 - \delta_1 k_1^2 & h_4 & h_1 - \delta_2 k_1^2 - i\kappa k_1 & -1 \\ -1 - i\alpha k_1 & 0 & -i\beta k_1 & -1 \end{pmatrix}. \tag{3.8}$$

Nontrivial solutions $\vec{X}_1^{(1)}$ exist for

$$\det\hat{\Omega}(k_1,\omega_1) = 0 \Rightarrow \omega_1 = \omega_1(k_1), \quad \omega_1 \in \mathbb{C}.$$

The complex valued function $\omega_1(k_1)$ is the dispersion relation of the linear theory and the instability curves (see Fig. 1) result from the condition $\mathrm{Im}(\omega_1) = 0$. Note that ω_1 depends also on the control parameter (3.1) through η_k, h_l, σ_m, ρ_n, and D. The corresponding eigenvector is given by

$$\vec{X}_1^{(1)} = \Phi(\xi,\tau)\vec{Y}^{(1)}, \quad \vec{Y}^{(1)} = \begin{pmatrix} 1 \\ M_{10} \\ V_{10} \\ W_{10} \end{pmatrix}, \tag{3.9}$$

where $\Phi(\xi,\tau)$ is an arbitrary amplitude depending on the stretched variables and the components of $\vec{Y}^{(1)}$ can be calculated by means of

$$M_{10} = \frac{-i\omega_1 - A(k_1)}{B(k_1)},$$

$$V_{10} = \frac{h_4 M_{10} + 2 + i\alpha k_1 - \delta_1 k_1^2}{\delta_2 k_1^2 + i\delta_4 k_1 - h_1}, \tag{3.10}$$

$$W_{10} = -1 - i\alpha k_1 - i\beta k_1 V_{10}.$$

Here $A(k_1)$ and $B(k_1)$ are defined by

$$A(k_1) = \eta_1 - \alpha k_1^2 + \frac{(\eta_2 - \beta k_1^2)(2 + i\alpha k_1 - \delta_1 k_1^2)}{\delta_2 k_1^2 + i\delta_4 k_1 - h_1},$$

$$B(k_1) = \eta_3 + \frac{h_4(\eta_2 - \beta k_1^2)}{\delta_2 k_1^2 + i\delta_4 k_1 - h_1}. \tag{3.11}$$

For the modes $\vec{X}_n^{(1)}$ with $n \geqslant 2$ one obtains similar systems as Eq. (3.7), but now with a coefficient matrix $\hat{\Omega}(k_n, \omega_n)$. Since $\hat{\Omega}(k_n,\omega_n) = \hat{\Omega}(nk_1, n\omega_1)$ the determinant of this matrix cannot be equal to zero. This means that $\vec{X}_n^{(1)}$ for $n \geqslant 2$ must be the trivial solution. The analysis of the $n = 0$ mode requires special consideration because the integral (2.5) does not vanish. In [7] we have shown that the $n = 0$ mode is stable for a sufficiently large external resistance R_a. Therefore, we set $\vec{X}_0^{(1)} = \vec{0}$ as well. The first-order perturbation theory must be completed by the adjoint homogeneous problem. Taking into account the usual L_2 scalar product of four-dimensional vectors

$$\langle \vec{Y} | \vec{X} \rangle = \frac{1}{l'} \int_0^{l'} \vec{Y}^\dagger(\xi)\vec{X}(\xi)\,d\xi,$$

where the dagger denotes the adjoint vector $[\vec{Y}^{\dagger}=(\vec{Y}^{T})^{*}]$ and $l'=\varepsilon l$. The adjoint linear problem is

$$\hat{\Omega}^{\dagger}\vec{V}=0. \tag{3.12}$$

Its solution $\vec{V}$ is easy to find as

$$\vec{V}=\Psi(\xi,\tau)\vec{Y}_{ad}^{(1)}, \quad \vec{Y}_{ad}^{(1)}=\begin{pmatrix}1\\ \tilde{M}_{10}\\ \tilde{V}_{10}\\ \tilde{W}_{10}\end{pmatrix}, \tag{3.13}$$

with the components

$$\tilde{M}_{10}=\frac{i\omega_1-A^{*}(k_1)}{C^{*}(k_1)}, \quad \tilde{V}_{10}=-\frac{\eta_5\tilde{M}_{10}+\eta_2-\beta k_1^2}{\delta_2k_1^2-i\delta_4k_1-h_1},$$

$$\tilde{W}_{10}=-\tilde{V}_{10}.$$

$A(k_1)$ is defined by Eq. (3.11) and $C(k_1)$ can be written as

$$C(k_1)=\eta_4+\frac{\eta_5(2+i\alpha k_1-\delta_1k_1^2)}{\delta_2k_1^2+i\delta_4k_1-h_1}.$$

In Eq. (3.13) the amplitude $\Psi(\xi,\tau)$ is an arbitrary function of the stretched variables.

B. Second order

To this order the mode numbers $n\geqslant 3$ provide homogeneous systems that have only the trivial solution

$$\vec{X}_n^{(2)}=\vec{0}, \quad n\geqslant 3.$$

On the other hand, we obtain an inhomogeneous equation for $n=2$,

$$\hat{\Omega}(2k_1,2\omega_1)\vec{X}_2^{(2)}=\Phi^2\vec{F}_2(X^{(1)}), \tag{3.14}$$

where $\hat{\Omega}$ is the same matrix as in Eq. (3.8), but now with the substitution $k_1\to 2k_1$ and $\omega_1\to 2\omega_1$. The inhomogeneity $\vec{F}_2$ depends on the first-order solutions. The explicit form of the components is given in the Appendix. The ansatz $\vec{X}_2^{(2)}=\Phi^2\vec{Y}_2^{(2)}$ reduces Eq. (3.14) to

$$\hat{\Omega}(2k_1,2\omega_1)\vec{Y}_2^{(2)}=\vec{F}_2. \tag{3.15}$$

Because $\det\hat{\Omega}(2k_1,2\omega_1)\neq 0$, this system can be solved by means of Kramer's rule.

Moreover, for $n=1$ one finds

$$\hat{\Omega}(k_1,\omega_1)\vec{X}_1^{(2)}=i\left(c\frac{\partial\hat{\Omega}}{\partial\omega_1}+\frac{\partial\hat{\Omega}}{\partial k_1}\right)\frac{\partial}{\partial\xi}\vec{X}_1^{(1)}. \tag{3.16}$$

A comparison with the first-order equation (3.7) shows that the homogeneous problem associated with Eq. (3.16) always has a nontrivial solution. Consequently, the inhomogeneous problem has nontrivial solutions if a Fredholm alternative condition is satisfied. Since the adjoint solution is given by Eq. (3.13), the solvability condition for Eq. (3.16) can be written by means of the first-order solutions

$$0=\frac{1}{l'}\int_0^{l'}\Psi^{*}\frac{\partial\Phi}{\partial\xi}d\xi\left\langle \vec{Y}_{ad}^{(1)}\middle| i\left(c\frac{\partial\hat{\Omega}}{\partial\omega_1}+\frac{\partial\hat{\Omega}}{\partial k_1}\right)\vec{Y}^{(1)}\right\rangle,$$

where the second factor $\langle\,|\,\rangle$ now is an usual Hermitian scalar product because $\vec{Y}_{ad}^{(1)}$, $\vec{Y}^{(1)}$, and $\hat{\Omega}$ do not depend on the stretched variables. In any case the integral will not be equal to zero because Ψ is an arbitrary function. Therefore, the scalar factor must be zero

$$0=\left\langle \vec{Y}_{ad}^{(1)}\middle| i\left(c\frac{\partial\hat{\Omega}}{\partial\omega_1}+\frac{\partial\hat{\Omega}}{\partial k_1}\right)\vec{Y}^{(1)}\right\rangle.$$

Taking into consideration the formula

$$0=\hat{\Omega}\frac{\partial}{\partial k_1}\vec{Y}^{(1)}+\left(\frac{\partial\omega_1}{\partial k_1}\frac{\partial\hat{\Omega}}{\partial\omega_1}+\frac{\partial\hat{\Omega}}{\partial k_1}\right)\vec{Y}^{(1)}, \tag{3.17}$$

which can be found by differentiation of the first-order equation $\hat{\Omega}\vec{Y}^{(1)}=\vec{0}$ with respect to k_1, one obtains

$$0=\left(c-\frac{\partial\omega_1}{\partial k_1}\right)\langle\vec{Y}_{ad}^{(1)}|\hat{T}\vec{Y}^{(1)}\rangle,$$

where $\partial\hat{\Omega}/\partial\omega_1=-i\hat{T}$ is used and $\hat{T}$ is the projector in Eq. (2.3). It is easy to show that the scalar product does not vanish near the instability curve and therefore

$$c=\frac{\partial\omega_1}{\partial k_1}=v_g, \tag{3.18}$$

i.e., the free parameter c introduced in Eq. (3.4) must be the group velocity of the wave at the critical point. Inserting Eq. (3.18) into Eq. (3.16), the explicit solution of the inhomogeneous problem (3.16) is easy to find by taking into account Eq. (3.17),

$$\vec{X}_1^{(2)}=-i\frac{\partial\Phi}{\partial\xi}\frac{\partial}{\partial k_1}\vec{Y}^{(1)}. \tag{3.19}$$

It must be emphasized that this is not the most general solution of the inhomogeneous problem because an arbitrary term $\tilde{\Phi}\vec{Y}^{(1)}$ that solves the homogeneous equation can be added. However, the "new" amplitude $\tilde{\Phi}$ cannot be fixed in our perturbation theory up to third order. Nevertheless, the term is important in a higher-order perturbation theory, which we plan to show in a forthcoming paper.

The last equation of second order is related to the $n=0$ mode. We find

$$\hat{\Omega}(0,0)\vec{X}_0^{(2)}+\frac{R}{R_a}\frac{\vec{n}}{l'}\int_0^{l'}w_0^{(2)}d\xi=\Phi^{*}\Phi\vec{F}_0(Y^{(1)}), \tag{3.20}$$

where

$$\vec{X}_0^{(2)}=\begin{pmatrix}u_0^{(2)}\\ m_0^{(2)}\\ v_0^{(2)}\\ w_0^{(2)}\end{pmatrix}, \quad \vec{n}=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix},$$

and the components of the inhomogeneity $\vec{F}_0$ can be found in the Appendix. In order to solve Eq. (3.20) we make the ansatz

$$\vec{X}_0^{(2)}=\Phi^*\Phi\vec{Y}_0^a+\frac{1}{l'}\int_0^{l'}\Phi^*\Phi\ d\xi\ \vec{Y}_0^b, \tag{3.21}$$

where the following vector components are used:

$$\vec{Y}_0^a=\begin{pmatrix} u_{11}^a \\ m_{11}^a \\ v_{11}^a \\ w_{11}^a \end{pmatrix},\quad \vec{Y}_0^b=\begin{pmatrix} u_{11}^b \\ m_{11}^b \\ v_{11}^b \\ w_{11}^b \end{pmatrix}. \tag{3.22}$$

Inserting the ansatz into Eq. (3.20) yields two systems of inhomogenous equations for the determination of $\vec{Y}_0^a$ and $\vec{Y}_0^b$ as

$$\hat{\Omega}(0,0)\vec{Y}_0^a=\vec{F}_0,\quad \hat{\Omega}(0,0)=\begin{pmatrix} -\eta_1 & -\eta_3 & -\eta_2 & 0 \\ -\eta_4 & -\eta_6 & -\eta_5 & 0 \\ 1 & h_4 & h_1 & -1 \\ -1 & 0 & 0 & -1 \end{pmatrix}, \tag{3.23}$$

and

$$\hat{\Lambda}\vec{Y}_0^b=-\frac{R}{R_a}w_{11}^a\vec{n},$$

$$\hat{\Lambda}=\begin{pmatrix} -\eta_1 & -\eta_3 & -\eta_2 & 0 \\ -\eta_4 & -\eta_6 & -\eta_5 & 0 \\ 1 & h_4 & h_1 & -1+R/R_a \\ -1 & 0 & 0 & -1-R/R_a \end{pmatrix}. \tag{3.24}$$

Because $\det\hat{\Omega}(0,0)\neq 0$, the solution of Eq. (3.23) can be found by means of Kramer's rule. The component w_{11}^a forms the inhomogeneity of the second system (3.24) and this system can be solved by a standard method too. Note that the matrix $\hat{\Lambda}$ depends on the external resistance R_a and there is the possibility that $\det\hat{\Lambda}$ has zeros at selected values of R_a. Indeed, such behavior is observed and we report on this in Sec. V. Moreover, in the limit $R_a\to\infty$ Eq. (3.24) degenerates to a homogeneous problem that only has the trivial solution

$$\lim_{R_a\to\infty}\vec{Y}_0^b=\vec{0}.$$

C. Third order

In this order it is sufficient to examine the $n=1$ mode only. We find

$$\hat{\Omega}(k_1,\omega_1)\vec{X}_1^{(3)}=-\hat{T}\frac{\partial}{\partial\tau}\vec{X}_1^{(1)}+\frac{1}{2}\frac{\partial^2\hat{\Omega}}{\partial k_1^2}\frac{\partial^2}{\partial\xi^2}\vec{X}_1^{(1)}+\hat{P}\vec{X}_1^{(1)}+\hat{T}v_g\frac{\partial}{\partial\xi}\vec{X}_1^{(2)}+i\frac{\partial\hat{\Omega}}{\partial k_1}\frac{\partial}{\partial\xi}\vec{X}_1^{(2)}+\Phi^*\Phi^2(\vec{E}+\vec{H})+\frac{\Phi}{l'}\int_0^{l'}\Phi^*\Phi\ d\xi\ \vec{G}, \tag{3.25}$$

where the vectors $\vec{E}=\vec{E}(Y^{(1)},\ Y^{(2)},Y_0^a)$, $\vec{H}=\vec{H}(Y^{(1)})$, and $\vec{G}=\vec{G}(Y^{(1)},Y_0^a,Y_0^b)$ are defined in the Appendix. $\hat{T}$ is the projector of Eq. (2.3) and $\hat{P}$ is the matrix given by

$$\hat{P}=-\frac{\partial}{\partial\varepsilon^2}[\hat{\Omega}(k_1,\omega_1)+i\omega_1\hat{T}]|_{\varepsilon=0,k_c}. \tag{3.26}$$

Since the corresponding homogeneous problem [cf. Eq. (3.7)] has a nontrivial solution, we use the Fredholm solvability condition as in Sec. III B. Inserting the solutions $\vec{X}_1^{(1)},\vec{X}_1^{(2)}$ on the right-hand side and using the adjoint solution (3.13), this condition reads

$$0=\frac{1}{l'}\int_0^{l'}\Psi^*\Bigg\{-\frac{\partial\Phi}{\partial\tau}\langle\vec{Y}_{ad}^{(1)}|\hat{T}\vec{Y}^{(1)}\rangle+\frac{1}{2}\frac{\partial^2\Phi}{\partial\xi^2}\left\langle\vec{Y}_{ad}^{(1)}\middle|\frac{\partial^2\hat{\Omega}}{\partial k_1^2}\vec{Y}^{(1)}\right\rangle-i\frac{\partial\omega_1}{\partial\varepsilon^2}\Phi\langle\vec{Y}_{ad}^{(1)}|\hat{T}\vec{Y}^{(1)}\rangle-i\frac{\partial^2\Phi}{\partial\xi^2}\left\langle\vec{Y}_{ad}^{(1)}\middle|v_g\hat{T}\frac{\partial}{\partial k_1}\vec{Y}^{(1)}\right\rangle+\frac{\partial^2\Phi}{\partial\xi^2}\left\langle\vec{Y}_{ad}^{(1)}\middle|\frac{\partial\hat{\Omega}}{\partial k_1}\frac{\partial}{\partial k_1}\vec{Y}^{(1)}\right\rangle+\Phi^*\Phi^2\langle\vec{Y}_{ad}^{(1)}|(\vec{E}+\vec{H})\rangle+\frac{\Phi}{l'}\int_0^{l'}\Phi^*\Phi\ d\xi\langle\vec{Y}_{ad}^{(1)}|\vec{G}\rangle\Bigg\}d\xi,$$

where we have used $\langle\vec{Y}_{ad}^{(1)}|(\partial\hat{\Omega}/\partial\varepsilon^2)\vec{Y}^{(1)}\rangle=0$, which can be proved by differentiation of the first-order equation $\hat{\Omega}\vec{Y}^{(1)}=0$. Since this equation must be true for all functions Ψ, we obtain

$$0=-\frac{\partial\Phi}{\partial\tau}+p\Phi+b\frac{\partial^2\Phi}{\partial\xi^2}+c\Phi^*\Phi^2+g\frac{\Phi}{l'}\int_0^{l'}\Phi^*\Phi\ d\xi, \tag{3.27}$$

where the coefficients are defined by

$$p=-i\frac{\partial\omega_1}{\partial\varepsilon^2}\bigg|_{\varepsilon=0,k_c},\quad b=\frac{i}{2}\frac{\partial^2\omega_1}{\partial k_1^2}\bigg|_{\varepsilon=0,k_c}$$

$$c=\frac{\langle\vec{Y}_{ad}^{(1)}|(\vec{E}+\vec{H})\rangle}{\langle\vec{Y}_{ad}^{(1)}|\hat{T}\vec{Y}^{(1)}\rangle},\quad g=\frac{\langle\vec{Y}_{ad}^{(1)}|\vec{G}\rangle}{\langle\vec{Y}_{ad}^{(1)}|\hat{T}\vec{Y}^{(1)}\rangle}. \tag{3.28}$$

Note that we have used some elementary manipulations that are based on differentiation of Eq. (3.17) with respect to k_1, taking into account the dispersion relation $\omega_1(k_1)$ to obtain the coefficient b. Of course, Eq. (3.27) is an amplitude equation of the Ginzburg-Landau type modified by an integral term. A similar term was found by Elmer [11] in a study concerning the nonlinear and nonlocal dynamics of spatially extended systems. Finally, we return to our original space and time variables and introduce the amplitude function

$$\varepsilon\Phi(\xi,\tau)=\tilde{A}(z,t)$$

to obtain

$$\frac{\partial\tilde{A}}{\partial t}=-v_g\frac{\partial\tilde{A}}{\partial z}+\varepsilon^2 p\tilde{A}+b\frac{\partial^2\tilde{A}}{\partial z^2}+c\tilde{A}^*\tilde{A}^2+g\frac{\tilde{A}}{l}\int_0^l\tilde{A}^*\tilde{A}\,dz. \tag{3.29}$$

It must be emphasized that the four coefficients p, b, c, and g are complex valued, whereas v_g is a real parameter. Let

$$p=p_r+ip_i,\quad b=b_r+ib_i,\quad c=c_r+ic_i,\quad g=g_r+ig_i, \tag{3.30}$$

where the index r means real part and i stands for the imaginary part of the complex coefficients. Then we can eliminate the terms $\sim p_i$ and $\sim g_i$ by means of a phase rotation

$$\tilde{A}(z,t)=\exp\left(i\varepsilon^2 p_i t+i\frac{g_i}{l}\int_0^t\int_0^l\tilde{A}^*\tilde{A}\,dz\,dt\right)A(z,t)$$

and this yields

$$\frac{\partial A}{\partial t}=-v_g\frac{\partial A}{\partial z}+\varepsilon^2 p_r A+(b_r+ib_i)\frac{\partial^2 A}{\partial z^2}+(c_r+ic_i)A^*A^2+g_r\frac{A}{l}\int_0^l A^*A\,dz. \tag{3.31}$$

Of course, one can eliminate the term $\sim v_g$ by means of a Galilei transformation. Moreover, scaling transformations are possible such that $b_r\to 1$, $c_r\to -1$, and $\varepsilon^2 p_r\to 1$. In order to discuss the limits $b_r\to 0$ and $c_r\to 0$, respectively, we have not performed this transformation, i.e., Eq. (3.31) is the form of the amplitude equation studied for the rest of this paper.

IV. WAVE SOLUTIONS AND THEIR STABILITY

This section deals with special stationary solutions of the amplitude equation (3.31) and with their stability. In particular, we look for wave solutions with constant amplitudes of the type

$$A(z,t)=R\ \exp\ i[(k-k_c)x+\phi_0], \tag{4.1}$$

with

$$x=z-(v_g+u)t, \tag{4.2}$$

where u and ϕ_0 are free real parameters and R is the constant wave amplitude. Note that u can be considered as the phase velocity of the plane wave (4.1) in the reference system moving with v_g. First we consider the trivial solution $R=0$, which corresponds to the homogeneous stationary positive column of the discharge. It is easy to show that this solution is unstable against perturbations of the form

$$\delta A=\delta A_0\exp[i(k-k_c)x-\lambda t]$$

if

$$\varepsilon^2 p_r>b_r(k-k_c)^2. \tag{4.3}$$

Of course, the instability curve $\varepsilon^2 p_r=b_r(k-k_c)^2$ reproduces the results of the linear theory near the critical point (cf. Sec. II).

A nontrivial wave solution with $R\neq 0$ is obtained by inserting Eq. (4.1) into the amplitude equation (3.31). We find

$$R^2=\frac{\varepsilon^2 p_r-b_r(k-k_c)^2}{-(c_r+g_r)}, \tag{4.4}$$

$$u=\frac{\varepsilon^2 p_r c_i+[b_i(c_r+g_r)-b_r c_i](k-k_c)^2}{(k-k_c)(c_r+g_r)}, \tag{4.5}$$

i.e., the amplitude R depends on the wave number, which clearly shows the nonlinear character of this solution. A real amplitude exists only if a supercritical bifurcation $(c_r+g_r<0)$ takes place at the critical point (cf. [7]). In this case the amplitude tends to zero at the neutral curve. Moreover, the wave propagates with a velocity different from the group velocity v_g as a consequence of the complex coefficients of the amplitude equation. For real coefficients $(c_i,b_i\to 0)$ the difference vanishes $(u\to 0)$. From the experimental point of view only stable waves can be observed and therefore we must study this property of our solution. The stability against small perturbations is described by the variational equation

$$\frac{\partial\delta A}{\partial t}=\varepsilon^2 p_r\delta A+u\frac{\partial\delta A}{\partial x}+(b_r+ib_i)\frac{\partial^2\delta A}{\partial x^2}+(c_r+ic_i)(A^2\delta A^*+2A^*A\delta A)+g_r\left(\frac{\delta A}{l}\int_0^l A^*A\,dx+\frac{A}{l}\int_0^l(A\delta A^*+A^*\delta A)dx\right),$$

where $\delta A(x,t)$ represents the small perturbation. Inserting the solution (4.1) for A and taking into consideration the complex conjugate variational equation by introducing the complex two-dimensional vector

$$\vec{S}=\begin{pmatrix}\delta A\\ \delta A^*\end{pmatrix},$$

we obtain

$$\frac{\partial\vec{S}_1}{\partial t}=\left(b_r\frac{\partial^2}{\partial x^2}-(c_r+g_r)R^2+[u-2b_i(k-k_c)]\frac{\partial}{\partial x}\right)\vec{S}_1+R^2\hat{C}\vec{S}_1+\left(b_i\frac{\partial^2}{\partial x^2}-c_iR^2+2b_r(k-k_c)\frac{\partial}{\partial x}\right)\hat{K}\vec{S}_1+R^2\hat{G}\frac{1}{l}\int_0^l\vec{S}_1dx, \tag{4.6}$$

where the vectors $\vec{S}$ and $\vec{S_1}$ are connected by a unitary transformation $\hat{U}(\phi)$,

$$\vec{S}_1=\hat{U}^{-1}(\phi)\vec{S},$$

$$\hat{U}=\frac{1}{\sqrt{2}}\begin{pmatrix}\exp(i\phi) & i\exp(i\phi)\\ \exp(-i\phi) & -i\exp(-i\phi)\end{pmatrix},\quad \phi=\phi_0+(k-k_c)x,$$

which depends on the phase of Eq. (4.1). The matrix operators $\hat{K}$, $\hat{C}$, and $\hat{G}$ are defined by

$$\hat{K}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \hat{C}=\begin{pmatrix}3c_r+g_r & -c_i\\ 3c_i & c_r+g_r\end{pmatrix},$$

$$\hat{G}=2\begin{pmatrix}g_r & 0\\ 0 & 0\end{pmatrix}.$$

In order to find the solution of Eq. (4.6), we make the ansatz

$$\vec{S}_1(x,t)=e^{\lambda t}[\vec{X}\cos(qx)+\vec{Y}\sin(qx)], \tag{4.7}$$

where q is an arbitrary wave number $q\neq k-k_c$ and $\vec{X},\vec{Y}$ are constant two-dimensional vectors. Because

$$\frac{1}{l}\int_0^l \vec{S}_1 dx=\begin{cases}0 & \text{if} \quad q\neq 0\\ e^{\lambda t}\vec{X} & \text{if} \quad q=0,\end{cases}$$

we have to distinguish between uniform $(q=0)$ and nonuniform perturbations $(q\neq 0)$, respectively. For $q=0$ we get the eigenvalues

$$\lambda_1=0,\quad \lambda_2=2(c_r+g_r)R^2 \tag{4.8}$$

and the corresponding eigenfunctions

$$\delta A_{\lambda_1}=ie^{i\phi},\quad \delta A_{\lambda_2}=\frac{c_r+ic_i+g_r}{|c_r+ic_i+g_r|}e^{\lambda_2 t}e^{i\phi}. \tag{4.9}$$

δA_{λ_1} is a pure phase mode, well known as the Goldstone branch (cf. [22]), whereas δA_{λ_2} represents a mixed amplitude phase mode. In the limit $c_i\rightarrow 0$ the eigenfunction δA_{λ_2} becomes the amplitude mode discussed by Elmer [11].

For nonuniform perturbations $(q\neq 0)$ we find a quadratic equation with complex coefficients

$$\lambda^2+(a_1+ib_1)\lambda+(a_0+ib_0)=0, \tag{4.10}$$

where

$$\begin{aligned}
a_1&=2(b_r q^2-c_r R^2),\\
b_1&=2q[2b_i(k-k_c)-u],\\
a_0&=q^2\bigg((b_r^2+b_i^2)q^2-4b_r^2(k-k_c)^2\\
&\quad+\frac{b_r c_r+b_i c_i}{c_r}(a_1-2b_r q^2)\bigg)-\frac{b_1^2}{4},\\
b_0&=\frac{a_1 b_1}{2}+2q(k-k_c)\frac{b_r}{c_r}[2(b_r c_i-b_i c_r)q^2-c_i a_1].
\end{aligned} \tag{4.11}$$

A necessary condition for stability (Re $\lambda<0$) of the constant amplitude wave (4.1) is

$$a_1>0. \tag{4.12}$$

This condition also arises in the limit $q\rightarrow 0$ because

$$\lim_{q\to 0} a_1=-2c_r R^2,\quad \lim_{q\to 0} b_1=\lim_{q\to 0} a_0=\lim_{q\to 0} b_0=0$$

and the corresponding eigenvalues are

TABLE I. Possible bifurcations depending on the signs of the parameters c_r and c_r+g_r.

Case A	Case B	Case C	Case D
$c_r+g_r<0$	$c_r+g_r>0$	$c_r+g_r>0$	$c_r+g_r<0$
$c_r>0$	$c_r>0$	$c_r<0$	$c_r<0$
supercritical	subcritical	subcritical	supercritical

$$\lambda_1(q\rightarrow 0)=0,\quad \lambda_2(q\rightarrow 0)=2c_r R^2. \tag{4.13}$$

Thus we find the same discontinuity $\lambda_2(q=0)\neq\lambda_2(q\rightarrow 0)$ as discussed by Elmer [11], reflecting the fundamental difference between a partial integro-differential equation and a partial differential equation. Since the sign of a_1 is determined by the parameters c_r and (c_r+g_r) [cf. also Eq. (4.4)] in the limit $q\rightarrow 0$, we find the four qualitatively different regions listed in Table I.

The cases B and C connected with a subcritical bifurcation at the critical point need calculations up to fifth order to saturate the instability. We expect that the system would select a new stable state, which would be described by higher-order nonlinear terms in the amplitude equation, e.g., by a term of fifth power of the modulated amplitude. According to Eq. (4.13), the wave solutions would be unstable in case A; however, as Elmer [11] has shown, this is correct for an infinitely long system only. If the system has a finite length l, a lower limit of the wave number q exists by $q_{min}=2\pi/l$ and one finds a small window of stable solutions around the critical point.

In case D we use a general calculation similar to [23]. As a consequence of Eq. (4.12), the stability condition can be written by means of the cubic polynomial

$$\begin{aligned}
P(a_1)&=a_1^2\bigg[a_1\frac{b_r c_r+b_i c_i}{c_r}\\
&\quad+\left(b_i^2-b_r^2-2\frac{b_i b_r c_i}{c_r}\right)q^2-4b_r^2(k-k_c)^2\bigg]\\
&\quad-16(k-k_c)^2 b_r^2\left(q^2\frac{b_r c_i-b_i c_r}{c_r}-a_1\frac{c_i}{2c_r}\right)^2
\end{aligned} \tag{4.14}$$

as

$$P(a_1)>0\Leftrightarrow \text{stability}. \tag{4.15}$$

In case D the waves are stable against arbitrary long-wavelength disturbances (i.e., $q\rightarrow 0$) if

$$\begin{aligned}
\lim_{q\to 0}P(a_1)&=4c_r^2R^4\bigg[-2c_r R^2\frac{b_r c_r+b_i c_i}{c_r}\\
&\quad-4b_r^2(k-k_c)^2\left(1+\frac{c_i^2}{c_r^2}\right)\bigg]>0.
\end{aligned}$$

We finally find

$$\varepsilon^2 p_r>b_r\left[1+\frac{2b_r(c_r+g_r)}{b_r c_r+b_i c_i}\left(1+\frac{c_i^2}{c_r^2}\right)\right](k-k_c)^2. \tag{4.16}$$

We expect that for finite systems ($q\rightarrow q_{min}$) there are deviations from the condition (4.16). For such systems the polynomial (4.14) must be analyzed by numerical methods; however, we have not performed such calculations because we are mainly interested in long discharges. Equation (4.16) defines in (ε,k) space the band of Eckhaus stable wave solutions. In contrast to the classical Eckhaus result, where the curvature of the parabola at the critical point is three times the curvature of the neutral curve, here we have a variable curvature depending on the external circuit by g_r and on the plasma parameters by c_r, c_i, b_r, b_i. In the limit of vanishing imaginary parts one gets the result of Elmer [11] for an amplitude equation with real coefficients and, on the other hand, for $g_r\rightarrow 0$ (i.e., $R_a\rightarrow\infty$) Eq. (4.16) reduces to the stable band of the complex Ginzburg-Landau equation [24]. The stability band shrinks to zero if the denominator $b_rc_r+b_ic_i$ changes its sign for a fixed sign of c_r+g_r. This is the well known Benjamin-Feir instability region, which we shall characterize by the parameter

$$\Delta=\frac{b_rc_r+b_ic_i}{c_r}\leqslant 0, \tag{4.17}$$

where the equality describes the Benjamin-Feir boundary. In this section we have discussed the stability of the constant amplitude waves only. We expect that the amplitude equation (3.31) also has solitary wave solutions similar to the hole solution and the shock-type solution of the complex Ginzburg-Landau equation [25,26]. On the other hand, Elmer [11] has found solutions with spatially periodic amplitudes in the limit of real coefficients. In any case the solution manifold of Eq. (3.31) requires further investigations.

V. DISCUSSION

Compared to our basic equation (2.3), the amplitude equation (3.31) is much simpler to handle. Special nontrivial solutions and their stability can be discussed. Also numerical calculations can be executed very quickly and precisely. All the information on the special plasma system is contained within the complex coefficients [cf. Eq. (3.28) and the Appendix]. The dependence of these coefficients on the plasma parameters, e.g., on the pressure p_0 or on the discharge radius r_0, is not easy to discuss because some intermediate steps such as the solution of inhomogeneous systems (cf. Sec. II B) must be performed.

Inserting the ansatz

$$\tilde{A}(z,t)=A_0(t)\exp[i(k-k_c)(z-v_gt)]$$

into Eq. (3.29), one finds the Hopf normal form equation for a single oscillating mode $A_0(t)$, which contains no derivative and no integral term. The Hopf parameter (i.e., the coefficient of the nonlinear term $|A_0|^2A_0$) at the critical point is given by the sum $c+g$. We have calculated this parameter for a neon discharge of pressure $p_0=200$ Pa, tube radius $r_0=1$ cm, and length $L=50.3$ cm. A comparison with the results given in [7] yields the correct value of the Hopf parameter. Discussing the various coefficients in Eq. (3.31), we state that only g_r and g_i depend on the resistance R_a. Therefore, Fig. 2 shows the dependence of the nonlinear coefficients g_r and g_i on the external resistance R_a. Remember that the coefficient g_i can be removed by a simple transformation [cf. Eq. (3.31)]. As expected, g_r and g_i tend to zero as R_a increases. One notes two singular cases of g_r. First, at $R_a\approx 26$ kΩ we observe a discontinuity of g_r that arises from a codimension-2 bifurcation, which, of course, cannot be described by the amplitude equation (3.31). In this case not only does a wave solution (4.1) bifurcate from the equilibrium at $\varepsilon=0$, but also the homogeneous $n=0$ mode becomes unstable (cf. [7]). Indeed, the discontinuity arises from a zero of the determinant $\det\hat{\Lambda}$ [cf. Eq. (3.24)] and the matrix $\hat{\Lambda}$ just describes the stability of the $n=0$ mode [7]. In Sec. III A we have assumed that the $n=0$ mode is stable (i.e., $\vec{X}_0^{(1)}=\vec{0}$), which is realized for R/R_a sufficiently small. Therefore, the amplitude equation (3.31) is applicable only for parameters that are located on the right-hand side of the discontinuity.

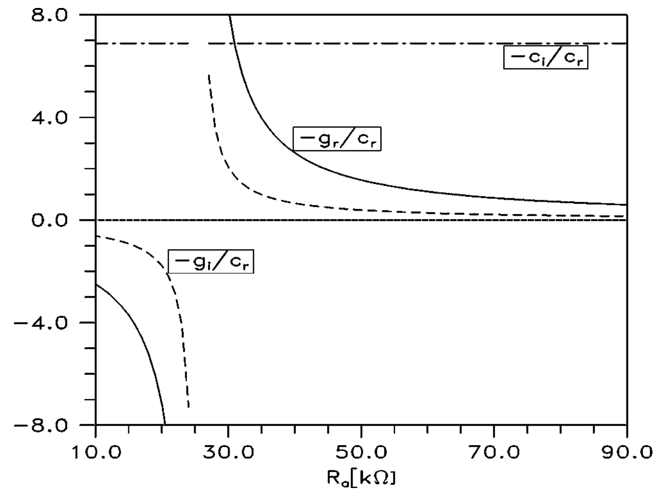

FIG. 2. Dependence of the nonlinear coefficients on the external resistance R_a. The parameters are $p_0=200$ Pa, $r_0=1$ cm, and $L=50.3$ cm.

The second singularity appears at $c_r+g_r=0$. Here the bifurcation changes from a supercritical to a subcritical one, i.e., the corresponding codimension-2 bifurcation is a generalized Hopf bifurcation (cf. [7]). This case is realized at $R_a\approx 65$ kΩ (cf. Fig. 2). In conclusion we note that the amplitude equation (3.31) may describe modulation phenomena near the instability border in the discussed plasma systems as long as the external resistance is sufficiently large, i.e., $g_r/-c_r\ll 1$. For all sufficiently small values of c_r+g_r [i.e., $c_r+g_r=O(\varepsilon^2)$] one has to taken into account additional terms in the amplitude equation, such as nonlinear gradient terms, a term with the fifth power of the modulated amplitude, and additional integral terms. The corresponding class of degenerate bifurcation problems is discussed by Eckhaus and Iooss [28]. The remarkable feature of this study is the phenomenon of strong pattern selection, which we have experimentally observed in a neon glow discharge as well [29]. The calculation of the additional terms in the degenerate case is in progress.

To discuss the influence of the global coupling term on the plasma system it is important to know the dependence of the coefficients in Eq. (3.31) on the plasma parameters. Using the equations of Sec. III, this task can be solved. Figure 3 shows the curve of the realized b_i/b_r and c_i/c_r values by

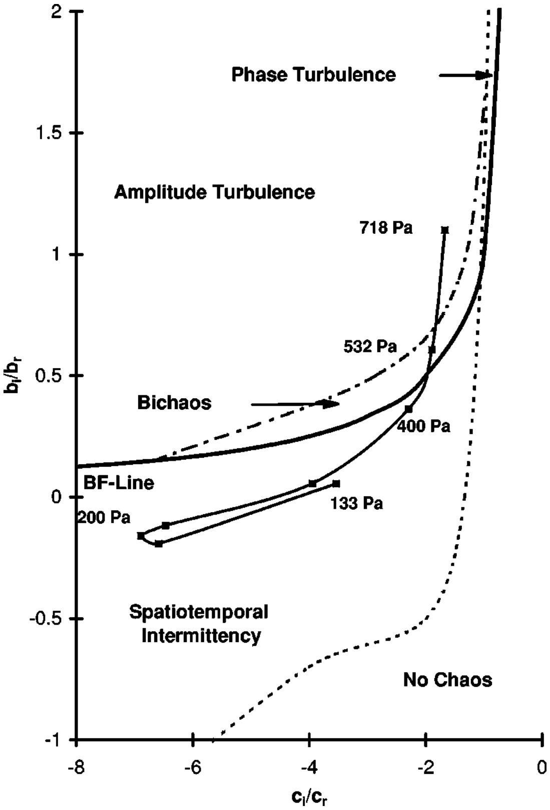

FIG. 3. Coefficients of the amplitude equation and the corresponding solutions for pressure variation in the interval from 133 Pa to 718 Pa ($r_0=1$ cm and the BF line is the Benjamin-Feir instability boundary).

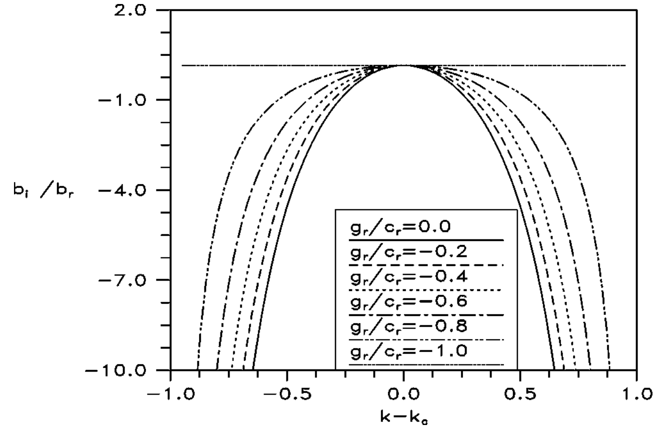

FIG. 4. Range of stable wave numbers and its variation with b_i/b_r for different values of g_r/c_r and fixed $c_i/c_r=-6.9$ and $\varepsilon^2 p_r/c_r=1$.

varying the gas pressure p_0 and holding the tube radius $r_0=1$ cm constant. To make the representation more comprehensible, we have added the known border lines separating the four regions of different behavior of the CGLE [27], i.e., in the case of $R_a\rightarrow\infty$ the intermittency, bichaos, and amplitude turbulence regimes are visited successively by enlarging the pressure.

In Sec. IV we have calculated the influence of the integral term on the properties of harmonic waves [see Eq. (4.1)]. According to Eqs. (4.4) and (4.5), a larger value of g_r enlarges the amplitude R and the phase velocity u. The latter is correct for sufficiently small wave numbers only. Whereas the Benjamin-Feir stability limit is not altered, the width of the Eckhaus range of stable wave numbers is magnified by a lowering of the external resistance. This can be seen in Fig. 4, where the gap of stable wave numbers and its variation with b_i/b_r is represented [cf. Eq. (4.16)]. Note that the global term in Eq. (1.1) preserves the intrinsic phase invariance of the CGLE. Probably this is the cause for the absence of the different phenomena observed in the case of other global terms [20].

To verify the most important assertions of this paper, we have made some numerical calculations starting from the basic equations (2.3). Strictly speaking, we made use of the completed form of Eq. (2.3), where the nonlinear production and loss terms are taken into account completely, i.e., without any power series expansion [7]. To facilitate the comparison with the theoretical predictions periodic boundary conditions were used. The length of the column was fixed at $L=50$ cm and the radius at $r_0=1$ cm, respectively. In all calculations the control parameter I_0 was chosen slightly above the critical current I_c. In particular the statements related to the appearance of turbulent behavior, which are based on the calculations of the coefficients of the Ginzburg-Landau equation (3.31) and are shown for different pressures in Fig. 3, are checked. Inspecting this figure, a very general prediction with respect to the full system (2.3) can be made: The behavior should be more turbulent if the pressure is raised. This is what is observed in experiments near the instability border [5]. This is also confirmed by the numerical results. For pressures of 133 Pa and 200 Pa, respectively, only traveling waves are observed. Starting at different initial conditions, the waves with the mode numbers $n=32$ (133 Pa) and $n=27$ (200 Pa), respectively, are the only attractors. On the other hand, at 718 Pa only irregular behavior is observed (cf. Fig. 5). Figure 5(a) shows the involved space-time diagram of the charge carrier density $u(z,t)$. To omit possible transient structures the diagram starts only at $t=15\,000$ and ends at $t=30\,000$. To show the irregular patterns more clearly and to compare with the behavior of Eq. (3.31) the complex modulation amplitude $A(z,t)$ is extracted with the help of the complex demodulation technique [30]. Furthermore, the data are transformed to a coordinate system moving with velocity $v_g=-0.291$, which is the group velocity at the critical point. In Figs. 5(b) and 5(c) the modulus and the phase gradient of $A(z,t)$ are represented. The zigzag motions of localized structures are clearly visible. Their velocity strongly deviates from the group velocity. In Fig. 5(c) these coherent structures are seen to be unstable. The phase gradient steepens until a phase slip (amplitude defect) appears and new left and right moving objects develop. These objects resemble the homoclinic holes investigated recently by van Hecke [26].

Looking at details, one observes serious differences between the qualitative behavior of solutions of Eq. (3.31) and the basic equations (2.3), respectively. For example, in the bichaos regime of Eq. (3.31) (i.e., at 532 Pa) only quasiperiodic solutions are observed. Amplitude turbulence and phase turbulence could not be detected in this regime. Furthermore, for lower pressures, as mentioned above, only traveling waves with one selected mode appear. No intermit-

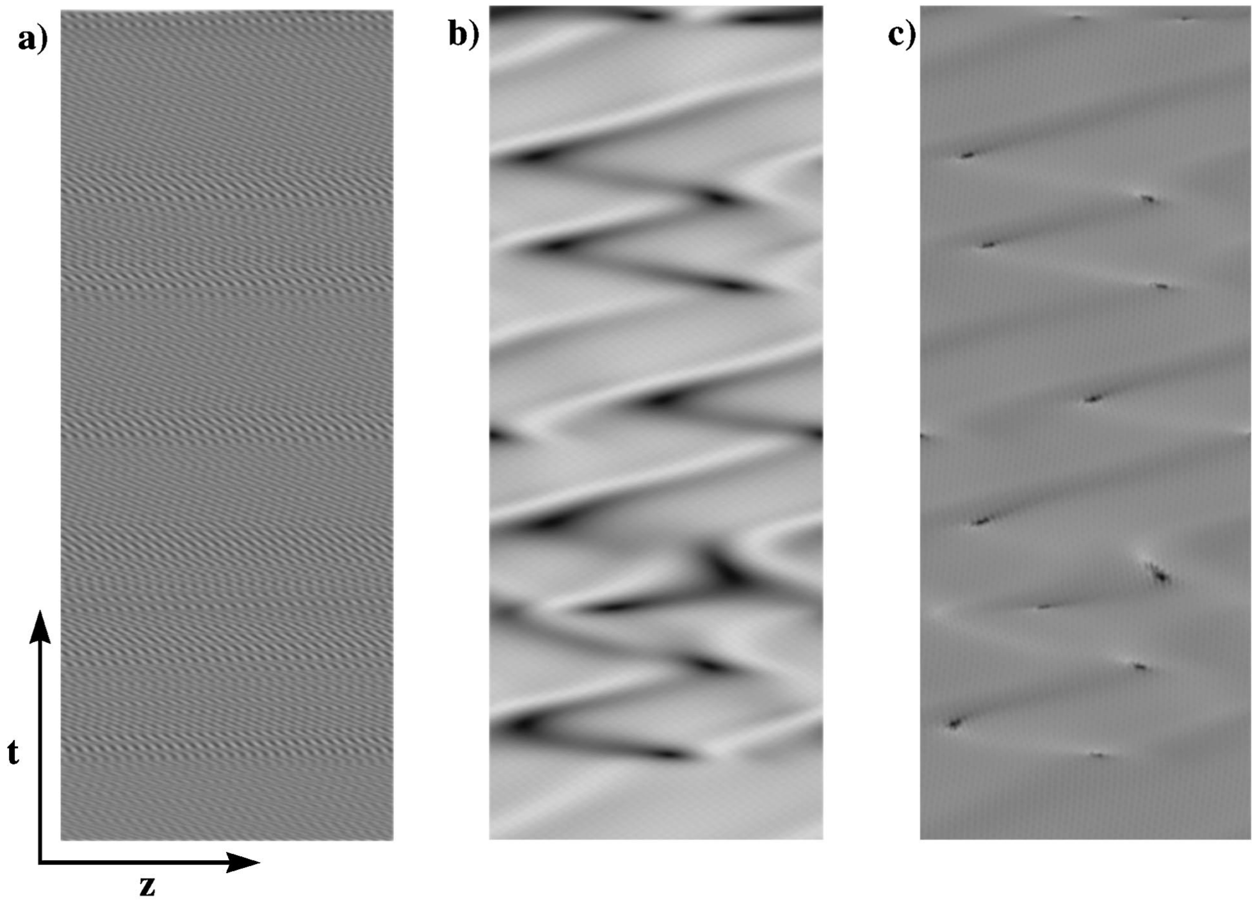

FIG. 5. Space-time diagram of the charge carrier density in different representations. The parameters are $p_0 = 720$ Pa, $I_0 = 0.778$ mA, $L = 50$ cm, and $r_0 = 1$ cm. (a) Density $u(z,t)$ and (b) and (c) modulus and phase gradient, respectively, of the data after complex demodulation and transformation to a system moving with v_g.

tency is found. These strong pattern selection [28] of traveling waves is also observed in the experiments in this pressure range [29]. We strongly suppose that the discrepancies observed can be diminished if terms of higher order, i.e., nonlinear gradient terms and the fifth-order term $|A|^4 A$, are considered. Moreover, we have done many numerical calculations at different pressures for short and medium lengths of the positive column. We plan to report on the results and their comparison with experiments in a forthcoming paper.

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APPENDIX: DEFINITIONS

In this appendix we list some of our abbreviations and definitions. We begin with the definition of the coefficients in Eq. (2.3),

$$\delta_1 = \epsilon(\gamma\alpha - \alpha^*), \quad \delta_2 = \epsilon(\gamma\beta - \beta^*),$$

$$\delta_3 = 2\delta_1 + \delta_2, \quad \delta_4 = \epsilon\gamma - \beta, \quad \kappa = \epsilon\gamma,$$

where

$$\alpha = 0.842, \quad \beta = 0.281, \quad \gamma = 1.476,$$

$$\alpha^* = 0.956, \quad \beta^* = 1.275, \quad \epsilon = 1.05.$$

In second-order perturbation theory we have used the vectors $\vec{F}_2$ and $\vec{F}_0$. The corresponding components are denoted by $F_2(j)$, with $j = 1,2,3,4$,

$$F_2(1) = [\rho_1 - 2(\alpha+\beta)k_1^2]V_{10} + \rho_2 V_{10}^2 + \rho_3 M_{10} + \rho_4 V_{10} M_{10} + \rho_5 M_{10}^2,$$

$$F_2(2) = \rho_6 V_{10} + \rho_7 V_{10}^2 + \rho_8 M_{10} + \rho_9 V_{10} M_{10} + \rho_{10} M_{10}^2, \tag{A1}$$

$$F_2(3) = [(2\delta_1 + \delta_2 + \delta_3)k_1^2 - h_1]V_{10} + (2\delta_2 k_1^2 - h_2)V_{10}^2 - h_4 M_{10} - h_5 V_{10} M_{10}$$

$$F_2(4) = i(\alpha+\beta)k_1 V_{10} + W_{10},$$

and for vector $\vec{F}_0$,

$$F_0(1)=\rho_1(V_{10}+V_{10}^*)+\rho_4(V_{10}M_{10}^*+V_{10}^*M_{10})+2\rho_2V_{10}^*V_{10}+\rho_3(M_{10}+M_{10}^*)+2\rho_5M_{10}M_{10}^*,$$

$$F_0(2)=\rho_6(V_{10}+V_{10}^*)+2\rho_7V_{10}^*V_{10}+\rho_8(M_{10}+M_{10}^*)+\rho_9(V_{10}M_{10}^*+V_{10}^*M_{10})+2\rho_{10}M_{10}^*M_{10},$$

(A2)

$$F_0(3)=-h_1(V_{10}+V_{10}^*)-2h_2V_{10}V_{10}^*-h_4(M_{10}+M_{10}^*)-h_5(V_{10}M_{10}^*+V_{10}^*M_{10}),$$

$$F_0(4)=ik_1(\alpha-\beta)(V_{10}^*-V_{10})+W_{10}+W_{10}^*.$$

Next, we list the vector components of $\vec{E}$, $\vec{H}$, and $\vec{G}$. These vectors are very important since they determine the nonlinear coefficients of the amplitude equation by Eq. (3.28). We denote the components of the solution vector $\vec{Y}_2^{(2)}$ in Eq. (3.15) by

$$\vec{Y}_2^{(2)}=\begin{pmatrix} u_{20}\\ m_{20}\\ v_{20}\\ w_{20}\end{pmatrix}.$$

Then, using Eqs. (3.9) and (3.22) one finds

$$E(1)=\alpha k_1^2(v_{20}-2V_{10}^*u_{20}-v_{11}^a)+\beta k_1^2(-2v_{20}+V_{10}^*u_{20}-u_{11}^aV_{10})+\rho_1(v_{11}^a+u_{11}^aV_{10}+u_{20}V_{10}^*+v_{20})+2\rho_2(V_{10}v_{11}^a+V_{10}^*v_{20})+2\rho_5(M_{10}m_{11}^a+M_{10}^*m_{20})+\rho_3(m_{11}^a+u_{11}^aM_{10}+u_{20}M_{10}^*+m_{20})+\rho_4(V_{10}m_{11}^a+v_{11}^aM_{10}+v_{20}M_{10}^*+V_{10}^*m_{20}),$$

$$E(2)=\rho_6(v_{11}^a+V_{10}u_{11}^a+u_{20}V_{10}^*+v_{20})+2\rho_7(V_{10}^*v_{20}+V_{10}v_{11}^a)+\rho_8(M_{10}^*u_{20}+m_{20}+m_{11}^a+M_{10}u_{11}^a)+2\rho_{10}(M_{10}m_{11}^a+m_{20}M_{10}^*)+\rho_9(V_{10}m_{11}^a+v_{11}^aM_{10}+M_{10}^*v_{20}+V_{10}^*m_{20}),$$

$$E(3)=2\delta_1k_1^2(v_{11}^a+v_{20}+4V_{10}^*u_{20})-2\delta_3k_1^2(v_{20}+u_{20}V_{10}^*)+\delta_2k_1^2(4v_{20}+u_{20}V_{10}^*+u_{11}^aV_{10})+(\delta_2k_1^2-2h_2)(V_{10}v_{11}^a+V_{10}^*v_{20})-h_1(v_{11}^a+V_{10}u_{11}^a+V_{10}^*u_{20}+v_{20})-h_4(m_{11}^a+M_{10}u_{11}^a+u_{20}M_{10}^*+m_{20})-h_5(v_{20}M_{10}^*+V_{10}^*m_{20}+V_{10}m_{11}^a+M_{10}v_{11}^a),$$

$$E(4)=i\alpha k_1(v_{11}^a-v_{20}+2V_{10}^*u_{20})+i\beta k_1(u_{11}^aV_{10}+2v_{20}-u_{20}V_{10}^*)+w_{11}^a+W_{10}u_{11}^a+w_{20}+u_{20}W_{10}^*$$

and the components of $\vec{H}$ are given by

$$H(1)=\sigma_0(M_{10}V_{10}^*+M_{10}^*V_{10}+M_{10}V_{10})+\sigma_1(2V_{10}^*V_{10}+V_{10}^2)+3\sigma_2V_{10}^2V_{10}^*+\sigma_3(2M_{10}V_{10}^*V_{10}+M_{10}^*V_{10}^2)+\sigma_4(2V_{10}M_{10}^*M_{10}+V_{10}^*M_{10}^2),$$

$$H(2)=\sigma_5(2V_{10}^*V_{10}+V_{10}^2)+\sigma_7(2M_{10}V_{10}^*V_{10}+M_{10}^*V_{10}^2)+3\sigma_6V_{10}^2V_{10}^*+\sigma_8(2V_{10}M_{10}^*M_{10}+V_{10}^*M_{10}^2),$$

$$H(3)=(\delta_1k_1^2-h_2)(2V_{10}V_{10}^*+V_{10}^2)-\delta_3k_1^2V_{10}^2-h_5(V_{10}M_{10}^*+V_{10}^*M_{10}+V_{10}M_{10})+2\delta_2k_1^2V_{10}^2-h_6(2M_{10}V_{10}V_{10}^*+M_{10}^*V_{10}^2)-3h_3V_{10}^2V_{10}^*,$$

$$H(4)=0.$$

Moreover, for $\vec{G}$ one finds

$$G(1)=\rho_1(v_{11}^b+u_{11}^bV_{10})+2\rho_2V_{10}v_{11}^b+\rho_3(m_{11}^b+u_{11}^bM_{10})+\rho_4(V_{10}m_{11}^b+v_{11}^bM_{10})+2\rho_5M_{10}m_{11}^b-\alpha k_1^2v_{11}^b-\beta k_1^2V_{10}u_{11}^b,$$

$$G(2)=\rho_6(v_{11}^b+V_{10}u_{11}^b)+2\rho_7V_{10}v_{11}^b+\rho_8(m_{11}^b+M_{10}u_{11}^b)+\rho_9(V_{10}m_{11}^b+M_{10}v_{11}^b)+2\rho_{10}M_{10}m_{11}^b,$$

$$G(3)=2\delta_1k_1^2v_{11}^b+\delta_2k_1^2V_{10}(u_{11}^b+v_{11}^b)-h_1(v_{11}^b+V_{10}u_{11}^b)-2h_2V_{10}v_{11}^b-h_4(m_{11}^b+M_{10}u_{11}^b)-h_5(V_{10}m_{11}^b+M_{10}v_{11}^b)-\frac{R}{R_a}(w_{11}^a+w_{11}^b)(W_{10}+i\kappa k_1V_{10}),$$

$$G(4)=i\alpha k_1v_{11}^b+i\beta k_1u_{11}^bV_{10}+w_{11}^b+W_{10}u_{11}^b.$$

Using the results of Sec. III B, it is easy to show that

$$\lim_{R_a\to\infty}\vec{G}=\vec{0}.$$

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